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Edge k-Product Cordial Labeling of Trees

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Abstract

The concepts of k-product cordial labeling and edge product cordial labeling were introduced in 2012 and further explored by various researchers. Building on these ideas, we define a new concept called 'edge k-product cordial labeling' as follows: For a graph $G = (V(G), E(G))$, which does not have isolated vertices, an edge labeling $f : E(G) \to \{0, 1, \dots, k-1\}$, where $k \geq 2$ is an integer, is said to be an edge k-product cordial labeling of G if it induces a vertex labeling $f^* : V(G) \to \{0, 1, \dots, k-1\}$ defined by $f^*(v) = \prod_{uv \in E(G)} f(uv) \pmod{k}$, which satisfies $\left|e_f(i) - e_f(j)\right| \leq 1$ and $\left|v_{f^*}(i) - v_{f^*}(j)\right| \leq 1$ for $i, j \in \{0, 1, \dots, k-1\}$, where $e_f(i)$ and $v_{f^*}(i)$ denote the number of edges and vertices, respectively, having label i for $i = 0, 1, \dots, k-1$. In this paper, we study the edge k-product cordial behavior of trees, a comet, and a double comet.

Keywords: cordial labeling; edge product cordial labeling; edge k-product cordial labeling; comet graph; double comet graph

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1. Preliminaries

The concept of graph labeling has experienced significant popularity over the past six decades, owing to its practical applications. A groundbreaking paper addressing graph labeling problems was published by Rosa [1]. Subsequently, numerous papers on various graph labeling methods have been published, and Gallian's survey [2] elegantly categorizes and organizes these diverse labeling methods published by various mathematicians all over the world. 'Cordial labeling' is one of the popular labelings introduced by Cahit [3]. Inspired by this notion, 'product cordial labeling' was proposed in [4]. In 2012, this concept was extended further, and a new concept called 'k-product cordial labeling' [5] was introduced. In the same year, Vaidya et al. [6] introduced a variation of product cordial labeling called 'edge product cordial labeling'. In this variant, the roles of vertices and edges in 'product cordial labeling' are interchanged. Since then, some more results on 'edge product cordial labeling' have been published by the same authors; see [6–9]. Building on this, Thamizharasi et al. [10] demonstrated the existence of 'edge product cordial labeling' of regular digraphs in 2015. In the subsequent years, Prajapati and Aboshady et al. [11–14] contributed additional results on 'edge product cordial labeling'. Motivated by the concepts of 'k-product cordial labeling' and 'edge product cordial labeling' and the established results, we put forth a new labeling, namely 'edge k-product cordial labeling', which extends the concept of edge product cordial labeling by expanding the set of labels from $\{0,1\}$ to $\{0,1,2,...,k-1\}$. Let $G=(V,E)$ be a graph without isolated vertices and $k \geq 2$. Let $f: E \to \{0,1,\dots,k-1\}$ be an edge labeling. The induced vertex labeling $f^*: V \to \{0,1,\dots,k-1\}$ is defined by

$$f^*(v) \equiv \prod_{uv \in E} f(uv) \pmod{k}.$$

f^* is called the induced labeling of f. For convenience, let $\mathbb{Z}_k = \{0,1,\dots,k-1\}$ be the complete residue system modulo k. Also, an edge is called *i-edge* if it is labeled by i; and a vertex is called *j-vertex* if its induced label is j. Let $e_f(i)$ and $v_{f^*}(i)$ denote the number of i-edges and i-vertices, respectively, for $i \in \mathbb{Z}_k$. f is said to be an *edge k-product cordial labeling* of G if $|e_f(i)-e_f(j)| \leq 1$ and $|v_{f^*}(i)-v_{f^*}(j)| \leq 1$ for $i,j \in \mathbb{Z}_k$. Also, G is called an *edge k-product cordial graph*.

Let L be a set of labels. For an edge labeling $f: E \to L$ of a graph $G=(V,E)$, if $|e_f(i)-e_f(j)| \leq 1$ for all $i,j \in L$, then we say that the edges of the graph labeled by labels in L *evenly* (under f).

In this paper, we use this notation, these concepts, and the following definitions unless otherwise stated.

Definition 1. *For $n \geq 1$, let $P_n = u_1 \cdots u_n$ be the path of order n. For $m \geq 1$, let the vertex set and the edge set of the star graph $K_{1,m}$ be $\{c\} \cup \{v_i \mid 1 \leq i \leq m\}$ and $\{cv_i \mid 1 \leq i \leq m\}$, respectively. For $m,n \geq 1$, let the graph $C(n,m) = P_n \cup K_{1,m}$ with identifying c with u_n. Such a graph is called a comet [15].*

Definition 2. *For $M \geq m \geq 1$, let the graph $DC(n,M,m) = P_n \cup K_{1,M} \cup K_{1,m}$ with the vertex set and the edge set $\{u_i \mid 1 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq M\} \cup \{w_i \mid 1 \leq i \leq m\}$ and $\{u_iu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_nv_i \mid 1 \leq i \leq M\} \cup \{u_1w_i \mid 1 \leq i \leq m\}$, respectively. Such a graph is called a double comet [16].*

Notation and concepts, which are not defined in this paper, are referred to in [17]. All graphs considered here are simple and connected.

We use the next section to show the structure of the subtree induced by all edges labeled by 0 of an edge k-product cordial tree. In the consecutive sections, we investigate the edge k-product cordial behavior of comet and double comet graphs for $k=3,4$, and 5.

2. Properties of Edge k-Product Cordial Trees

Lemma 1. *Let $f: E \to \mathbb{Z}_k$ be an edge labeling of a tree $T=(V,E)$, where $k \geq 2$. Then $v_{f^*}(0) \geq e_f(0)+1$, where f^* is the induced labeling of f.*

Proof. Let $E_0 = \{e \in E \mid f(e)=0\}$ and $T_0 = T[E_0]$, the edge induced subgraph of T. Let ω, p, and q be the number of components, the order and the size of T_0, respectively. Since T_0 is a forest, $p = q + \omega$. Since all its vertices are 0-vertices, $v_{f^*}(0) \geq p = q+\omega \geq e_f(0)+1$. □

Corollary 1. *Let f be an edge k-product cordial labeling of a tree T of order n, where $k \geq 2$. If E_0 is the set of 0 edges under f, then $T_0 = T[E_0]$ is a tree and*

$$v_{f^*}(0) = \begin{cases} \frac{n}{k} & \text{if } n \equiv 0 \pmod{k}; \\ \lfloor \frac{n}{k} \rfloor + 1 & \text{if } n \not\equiv 0 \pmod{k}, \end{cases} \quad e_f(0) = \begin{cases} \frac{n}{k} - 1 & \text{if } n \equiv 0 \pmod{k}; \\ \lfloor \frac{n}{k} \rfloor & \text{if } n \not\equiv 0 \pmod{k}. \end{cases}$$

Proof. For each i $(0 \leq i \leq k-1)$, we have

$$v_{f^*}(i) = \begin{cases} \frac{n}{k} & \text{if } n \equiv 0 \pmod{k}; \\ \lfloor \frac{n}{k} \rfloor \text{ or } \lfloor \frac{n}{k} \rfloor + 1 & \text{if } n \not\equiv 0 \pmod{k}, \end{cases}$$

$$e_f(i) = \begin{cases} \frac{n}{k} - 1 \text{ or } \frac{n}{k} & \text{if } n \equiv 0 \pmod{k}; \\ \frac{n-1}{k} = \lfloor \frac{n}{k} \rfloor & \text{if } n \equiv 1 \pmod{)}k \\ \lfloor \frac{n}{k} \rfloor \text{ or } \lfloor \frac{n}{k} \rfloor + 1 & \text{if } n \not\equiv 0,1 \pmod{k}. \end{cases}$$

By Lemma 1, we obtain this corollary. □

Example 1. *Suppose $n = 7$ and $k = 3$. Then T contains 7 vertices and 6 edges. We keep all notation defined in the proof of Lemma 1 and Corollary 2. For any edge 3-product cordial labeling f, $e_f(0) = e_f(1) = e_f(2) = 2$ and $v_{f^*}(i) = 2,3$ for $i = 0,1,2$. From the proof of Lemma 1, $v_{f^*}(0) \geq e_f(0) + \omega$. Since $2 \leq v_{f^*}(0) \leq 3$ and $e_f(0) + \omega = 2 + \omega \geq 3$, $\omega = 1$ and hence $v_{f^*}(0) = 3$.*

Lemma 2. *Let G be a graph of order k. If k is a prime, then G is not an edge k-product cordial graph.*

Proof. Let f be an edge k-product cordial labeling of G. Then $v_{f^*}(i) = 1$ for $i \in \mathbb{Z}_k$. In particular, $v_{f^*}(0) = 1$. Suppose there is a 0-edge, then it induces two 0-vertices, which is a contradiction. Suppose there are no 0-edges. Since k is a prime, there is no induced 0-vertex, which is also a contradiction. □

Corollary 2. *Let T be a tree of order k. If k is a prime or $k = 4$, then T is not an edge k-product cordial.*

Proof. When k is a prime, by Lemma 2, T is not edge k-product cordial. For $k = 4$, let f be an edge 4-product cordial labeling of T. Then $e_f(i) = 0,1$ and $v_{f^*}(i) = 1$ for each i. Let u be the 0-vertex. By Lemma 2, there is no 0-edge in T. Thus, $e_f(i) = 1$ for $1 \leq i \leq 3$. This results in $v_{f^*}(2) = 2$, which is a contradiction. □

3. Edge 3-Product Cordial Trees

In order to establish the edge 3-product cordial behavior of the comet, we prove the following two lemmas.

Lemma 3. *The path graph P_n is an edge 3-product cordial for $n \geq 4$.*

Proof. We define an edge labeling f_n for P_n recursively, and this labeling is represented by $(a_1, \ldots, a_{n-1})$ if $f_n(u_i u_{i+1}) = a_i$, $1 \leq i \leq n-1$. We will use this representation for a labeling f_n of P_n in the whole paper.

The edge 3-product cordial labeling for P_n, $4 \leq n \leq 9$, is shown in Table 1.

Table 1. Edge 3-product cordial labeling for P_n, $4 \leq n \leq 9$.

n	f_n	$e_{f_n}(0)$	$e_{f_n}(1)$	$e_{f_n}(2)$	$v_{f_n^*}(0)$	$v_{f_n^*}(1)$	$v_{f_n^*}(2)$
4	(1,2,0)	1	1	1	2	1	1
5	(1,1,2,0)	1	2	1	2	2	1
6	(1,2,2,1,0)	1	2	2	2	2	2
7	(1,2,2,1,0,0)	2	2	2	3	2	2
8	(1,2,2,1,1,0,0)	2	3	2	3	3	2
9	(1,2,2,1,1,2,0,0)	2	3	3	3	3	3

For $n \geq 10$, define f_n recurrently as $f_n = f_{m-i+6} = (1,2,2,1,a_1,\ldots,a_{m-i-1},0,0)$, where $0 \leq i \leq 5$ and $m \geq 9$. Note that $a_1 = 1$. Clearly, $e_{f_{m-i+6}}(j) = e_{f_{m-i}}(j) + 2$, $v_{f^*_{m-i+6}}(j) = v_{f^*_{m-i}}(j) + 2$, $0 \leq j \leq 2$, where $f_{m-i} = (a_1,\ldots,a_{m-i-1})$. Thus,

$$\begin{aligned} e_{f_n}(0) = e_{f_n}(1) = e_{f_n}(2) &= \left\lfloor \frac{n}{3} \right\rfloor, & v_{f_n^*}(0) - 1 = v_{f_n^*}(1) = v_{f_n^*}(2) &= \left\lfloor \frac{n}{3} \right\rfloor \text{ if } n \equiv 1 \pmod 3; \\ e_{f_n}(0) = e_{f_n}(1) - 1 = e_{f_n}(2) &= \left\lfloor \frac{n}{3} \right\rfloor, & v_{f_n^*}(0) - 1 = v_{f_n^*}(1) - 1 = v_{f_n^*}(2) &= \left\lfloor \frac{n}{3} \right\rfloor \text{ if } n \equiv 2 \pmod 3; \\ e_{f_n}(0) + 1 = e_{f_n}(1) = e_{f_n}(2) &= \frac{n}{3}, & v_{f_n^*}(0) = v_{f_n^*}(1) = v_{f_n^*}(2) &= \frac{n}{3} \text{ if } n \equiv 0 \pmod 3. \end{aligned}$$

Hence, f_n is an edge 3-product cordial labeling of P_n. Note that $f_n^*(u_1) = 1$ and $f^*(u_n) = 0$. □

From [18], we have the following result.

Lemma 4. *The star graph $K_{1,m}$ is an edge 3-product cordial for $m \geq 3$.*

Definition 3. *Suppose G and H are the two edge-disjoint graphs with edge labelings g and h, respectively. We say that ϕ is a combination of g and h (or combine g with h) if*

$$\phi(x) = \begin{cases} g(x) & \text{if } x \in E(G), \\ h(x) & \text{if } x \in E(H). \end{cases}$$

Theorem 1. *The comet graph $C(n,m)$ is edge 3-product cordial for $n \geq 3$ and $m \geq 2$.*

Proof. Note that $C(n,m) = P_n \cup K_{1,m}$. Let $c = u_n$. We label P_n by f_n, which was defined in the proof of Lemma 3. Define labeling g_m, $m = 2,3,4$, for $K_{1,m}$ as follows:

A. Suppose $n \equiv 1 \pmod 3$.
If $m = 2$, then $g_2(cv_1) = 1$, $g_2(cv_2) = 2$.
If $m = 3$, then $g_3(cv_1) = 1$, $g_3(cv_2) = 2$, $g_3(cv_3) = 0$.
If $m = 4$, then $g_4(cv_1) = 1$, $g_4(cv_2) = 2$, $g_4(cv_3) = 0$, $g_4(cv_4) = 1$.

B. Suppose $n \equiv 2 \pmod 3$.
If $m = 2$, then $g_2(cv_1) = 0$, $g_2(cv_2) = 2$.
If $m = 3$, then $g_3(cv_1) = 1$, $g_3(cv_2) = 2$, $g_3(cv_3) = 0$.
If $m = 4$, then $g_4(cv_1) = 1$, $g_4(cv_2) = 2$, $g_4(cv_3) = 0$, $g_4(cv_4) = 2$.

C. Suppose $n \equiv 0 \pmod 3$.
If $m = 2$, then $g_2(cv_1) = 1$, $g_2(cv_2) = 0$.
If $m = 3$, then $g_3(cv_1) = 1$, $g_3(cv_2) = 2$, $g_3(cv_3) = 0$.
If $m = 4$, then $g_4(cv_1) = 1$, $g_4(cv_2) = 2$, $g_4(cv_3) = 0$, $g_4(cv_4) = 0$.

Let ϕ be the combination of f_n and g_m. We can check that $e_\phi(1) \geq e_\phi(2) \geq e_\phi(0)$ and $e_\phi(1) - e_\phi(0) \leq 1$; $v_{\phi^*}(0) \geq v_{\phi^*}(1) \geq v_{\phi^*}(2)$ and $v_{\phi^*}(0) - v_{\phi^*}(2) \leq 1$.

Note that $C(n,m) = C(n,m-3) \cup K_{1,3}$ with the common vertex u_n. If ϕ is an edge 3-product cordial labeling of $C(n,m-3)$, $m \geq 5$, then combine ϕ for $C(n,m-3)$ and g_3 for $K_{1,3}$ to obtain an edge 3-product cordial labeling for $C(n,m)$.

This completes the proof. □

Example 2. *Here is an example to illustrate the proof of Theorem 1. Suppose $n = 5$ and $m = 7$. Firstly, we label $C(5,4)$. According to the labeling defined in the proof above, we have the following labeling (Figure 1):*

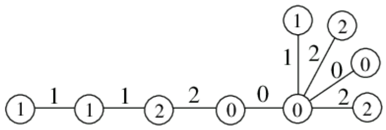

Figure 1. Edge 3-product cordial labeling of $C(5,4)$.

Here we can see that $e_\phi(0) = 2$, $e_\phi(1) = e_\phi(2) = 3$; $v_{\phi^*}(0) = v_{\phi^*}(1) = v_{\phi^*}(2) = 3$. Now, we combine the labeling g_3 of $K_{1,3}$ to the labeling ϕ of $C(5,4)$. We have (see Figure 2)

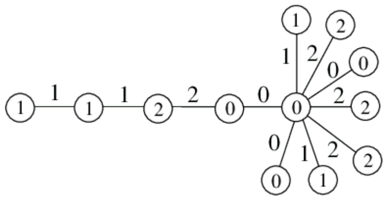

Figure 2. Edge 3-product cordial labeling of $C(5,7)$.

Here we can see that the number of 0-edges is 3, the number of i-edges is 4, $i = 1,2$, and the number of j-vertices is 4, $0 \leq j \leq 2$.

Remark 1. *By Lemma 3, $C(n,0) \cong P_n$ when $n \geq 4$ and $C(n,1) \cong P_{n+1}$ when $n \geq 3$ are edge 3-product cordial graphs. Again by Lemma 4, $C(1,m) \cong K_{1,m}$ for $m \geq 3$ and $C(2,m) \cong K_{1,m+1}$ for $m \geq 2$ are edge 3-product cordial graphs.*

Also, note that under the labeling defined in Lemma 3, the vertex u_1 is always a 1-vertex and u_n is always 0-vertex.

Consequently, if $C(n,m)$ is not isomorphic to P_1, P_2, or P_3, then $C(n,m)$ admits an edge 3-product cordial labeling ϕ such that $\phi(u_1) = 1$ and $\phi(u_n) = 0$.

Theorem 2. *The double comet graph $DC(n,M,m)$ is edge 3-product cordial for $n \geq 2$ and $M \geq m \geq 2$.*

Proof. Let $S_1 = K_{1,M}$ with the center u_n and $S_2 = K_{1,m}$ with the center u_1. Then $DC(n,M,m) = P_n \cup S_1 \cup S_2$.

For $2 \leq q \leq 5$, we label the edges of $S_2 = K_{1,q}$ and the selected edges of S_1 by 0,1,2 evenly as shown below and denote this labeling by α.

1. When $q = 2$, $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_nv_M) = \alpha(u_nv_{M-1}) = 0$ and $\alpha(u_nv_{M-2}) = \alpha(u_nv_{M-3}) = 2$.
2. When $q = 3$, $\alpha(u_1w_1) = 1$, $\alpha(u_1w_2) = \alpha(u_1w_3) = 2$, $\alpha(u_nv_M) = \alpha(u_nv_{M-1}) = 0$ and $\alpha(u_nv_{M-2}) = 1$.
3. When $q = 4$, $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_1w_3) = \alpha(u_1w_4) = 2$, $\alpha(u_nv_M) = \alpha(u_nv_{M-1}) = 0$.
4. When $q = 5$, $\alpha(u_1w_1) = \alpha(u_1w_2) = \alpha(u_1w_3) = 1$, $\alpha(u_1w_4) = \alpha(u_1w_5) = 2$, $\alpha(u_nv_M) = \alpha(u_nv_{M-1}) = \alpha(u_nv_{M-2}) = 0$ and $\alpha(u_nv_{M-3}) = 2$.

Note that $\alpha^*(u_1) = 1$ and $\alpha^*(u_n) = 0$. Also, $v_{\alpha^*}(0) = v_{\alpha^*}(1) = v_{\alpha^*}(2) + 1$.

Now, consider $m = 4p + q$, where $2 \leq q \leq 5$ and $p \geq 0$. We split S_2 into $K_{1,q}$ and $K_{1,4p}$ with the common vertex u_1. We label $K_{1,q}$, and the selected edges of S_1 by α as defined above, and all the edges of $K_{1,4p}$ by 1 and 2 evenly. Again, we label the edges of an unlabeled subgraph of S_1, which is isomorphic to $K_{1,2p}$ by 0. We denote this labeling by α. Then we have $\alpha^*(u_1) = 1$ and $\alpha^*(u_n) = 0$; $v_{\alpha^*}(0) = v_{\alpha^*}(1) = v_{\alpha^*}(2) + 1$; and all $e_\alpha(i)$ are the same for $0 \le i \le 2$.

Here, the unlabeled subgraph, say H, of $DC(n, M, m)$ is isomorphic to $C(n, M - 2p - \epsilon)$, where $\epsilon = 2, 3, 4$, which depends on q. That is,

1. When $q = 2$, $H \cong C(n, M - 2p - 4)$ if $M - 2p - 4 \ge 0$.
2. When $q = 3$, $H \cong C(n, M - 2p - 3)$.
3. When $q = 4$, $H \cong C(n, M - 2p - 2)$.
4. When $q = 5$, $H \cong C(n, M - 2p - 4)$.

If $M - 2p - \epsilon \ge 2$, then by Theorem 1 there exists an edge 3-product cordial labeling ϕ for H. Note that $v_{\alpha^*}(0) - 1 = v_{\alpha^*}(1) - 1 = v_{\alpha^*}(2)$, $\alpha^*(u_1) = 1$, $\alpha^*(u_n) = 0$ and $\phi^*(u_1) = 1$. When we combine α with ϕ, the number of j-vertices ($0 \le j \le 2$) are $v_{\phi^*}(0) + v_{\alpha^*}(0) - 1$, $v_{\phi^*}(1) + v_{\alpha^*}(1) - 1$ and $v_{\phi^*}(2) + v_{\alpha^*}(2)$. So the combination of α and ϕ results in an edge 3-product cordial labeling for $DC(n, M, m)$.

The detailed labeling for receiving edge 3-product cordial of $DC(n, M - 2p - \epsilon)$ for $M - 2p - \epsilon \le 1$ are moved to Appendix A. This completes the proof. □

Example 3. *The following Figure 3 illustrates the proof of Theorem 2.*

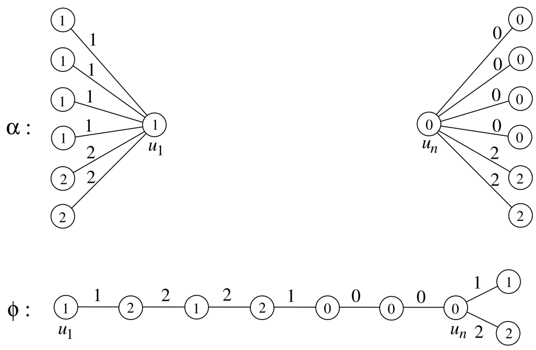

Figure 3. Illustration of edge 3-product cordial labeling of $DC(7, 8, 6)$.

We combine α and ϕ to get an edge 3-product cordial labeling for $DC(7, 8, 6)$.

4. Edge 4-Product Cordial Trees

We begin this section with the necessary condition on the number of vertices and leaves for a tree to admit an edge 4-product cordial labeling.

Theorem 3. *Let T be a tree with n vertices, and S be the set of all the leaves of T. If T is an edge 4-product cordial graph, then*

$$|S| \ge \begin{cases} \lfloor \frac{n}{4} \rfloor & \text{if } n \equiv 0, 1 \pmod 4, \\ \lfloor \frac{n}{4} \rfloor - 1 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Moreover, the bound is sharp.

Proof. Let f be an edge 4-product cordial labeling of T, and E_i be the set of edges labeled by i, where $i = 0, 2$. Let $T_i = T[E_i]$. By Corollary 1, T_0 is a tree and $v_{f^*}(0) = e_f(0) + 1 = |V(T_0)|$. This implies that each vertex in $V(T) \setminus V(T_0)$ is not labeled by 0.

Now, consider forest T_2. If $v \in V(T_2)$, then $f^*(v) = 0, 2$; and if $v \notin V(T_2)$, then $f^*(v) \ne 2$. If $v \in V(T_2)$ and $f^*(v) = 0$, then $v \in V(T_0) \cap V(T_2) = A_0$. If $f^*(v) = 2$, then $\deg_{T_2}(v) = 1$.

Let $u_1v_1, \ldots, u_qv_q$ be the edges in E_2 such that $u_i \in V(T_0)$ and $v_i \in V(T_2)$, $1 \le i \le q$. Note that v_i are distinct, but u_i may not be distinct. Then, there are $e_f(2) - q$ edges in E_2, and their end vertices are labeled by 2. Then $v_{f^*}(2) = q + 2(e_f(2) - q) = 2e_f(2) - q$, equivalently $q = 2e_f(2) - v_{f^*}(2)$.

(1) If $n \equiv 0 \pmod 4$, then $v_{f^*}(i) = \frac{n}{4}$ for all i and $e_f(0) = \frac{n}{4} - 1$. Thus, $e_f(2) = \frac{n}{4} = e_f(1) = e_f(3)$. Therefore, $q = \frac{n}{4}$.
(2) If $n \equiv 1 \pmod 4$, then $e_f(i) = \lfloor \frac{n}{4} \rfloor$ for all i, and $v_{f^*}(0) = \lfloor \frac{n}{4} \rfloor + 1$. Therefore, $v_{f^*}(i) = \lfloor \frac{n}{4} \rfloor$ for $i = 1, 2, 3$. Hence, $q = \lfloor \frac{n}{4} \rfloor$.
(3) If $n \equiv 2, 3 \pmod 4$, then $q = 2e_f(2) - v_{f^*}(2) \ge 2\lfloor \frac{n}{4} \rfloor - (\lfloor \frac{n}{4} \rfloor + 1) = \lfloor \frac{n}{4} \rfloor - 1$.

We merge the tree T_0 into a vertex r to receive the resultant graph T'. Then T' is a rooted tree with root r and $\deg_{T'}(r) = q$. So T' has at least q leaves, which are also the leaves of T. Since S is the set of all leaves of T and T contains at least q leaves, $|S| \ge q$. Hence,

$$|S| \ge \begin{cases} \lfloor \frac{n}{4} \rfloor & \text{if } n \equiv 0, 1 \pmod 4, \\ \lfloor \frac{n}{4} \rfloor - 1 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

□

The following remark demonstrates that the bound in Theorem 3 is sharp.

Remark 2. *If P_n is an edge 4-product cordial, then*

$$n \le \begin{cases} 9 & \text{if } n \equiv 0, 1 \pmod 4, \\ 15 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

Clearly, P_4, P_3, and P_2 are not edge 4-product cordial graphs. An edge 4-product cordial labeling for P_n, $5 \le n \le 15$, is shown in Table 2.

Table 2. Edge 4-product cordial labeling for P_n, $5 \le n \le 15$.

n	Label the Edges in Order	v(0)	v(1)	v(2)	v(3)
5	2,0,3,1	2	1	1	1
6	2,0,2,3,1	2	1	2	1
7	2,0,2,1,3,3	2	1	2	2
8	2,0,2,1,3,3,1	2	2	2	2
9	2,0,0,2,1,3,3,1	3	2	2	2
10	2,0,0,2,1,3,3,1,3	3	2	2	3
11	2,0,0,2,3,3,1,1,3,1	3	3	2	3
14	2,0,0,0,2,1,3,3,1,1,3,3,2	4	3	4	3
15	2,0,0,0,2,1,3,3,1,1,3,3,1,2	4	3	4	4

Corollary 3. *P_n is an edge 4-product cordial if and only if $n \in \{5, 6, 7, 8, 9, 10, 11, 14, 15\}$.*

Note that in the following lemmas and corollary, all the induced vertex labelings work in the complete residues class modulo 4, $\mathbb{Z}_4$.

Lemma 5. *Suppose S_1 and S_2 are the trees such that $V(S_1) \cap V(S_2) = \{x\}$. Let the order of S_1 and S_2 be a and b, respectively, such that $2n \le a \le b \le 2n + 1$ for some positive integer n. Let $f : E(S_1) \to \{0, 2\}$ and $g : E(S_2) \to \{1, 3\}$ be the edge labeling, which satisfy the following conditions:*

(1) $0 \le e_f(2) - e_f(0) \le 1$ *and* $0 \le v_{f^*}(2) - v_{f^*}(0) \le 1$;
(2) $|e_g(1) - e_g(3)| \le 1$ *and* $|v_{g^*}(1) - v_{g^*}(3)| \le 1$;
(3) $g^*(x) = \ell$ *only if* $v_{g^*}(\ell_1) \le v_{g^*}(\ell)$, *where* $\{\ell, \ell_1\} = \{1, 3\}$.

Let ϕ be the combination of f and g. Then ϕ is an edge 4-product cordial labeling of $S_1 \cup S_2$.

Proof. Suppose the order of S_1 is $2n$. Then the order of S_2 is either $2n$ or $2n+1$. Thus, by conditions 1 and 2, we obtain $|e_\phi(i) - e_\phi(j)| \leq 1$ for all $i \neq j$.

Since $f^*(x) = 0$ or 2, we have $\phi^*(x) = f^*(x)g^*(x) = f^*(x)$. Thus, the number of 0-vertices and 2-vertices does not change, and they are equal to n.

If the order of S_2 is $2n+1$ and $g^*(x) = 1$, then $v_{g^*}(1) = n+1$ and $v_{g^*}(3) = n$. Hence, $v_{\phi^*}(1) = n$ and $v_{\phi^*}(3) = n$.

If the order of S_2 is $2n+1$ and $g^*(x) = 3$, then $v_{g^*}(1) = n$ and $v_{g^*}(3) = n+1$. Hence, $v_{\phi^*}(1) = n$ and $v_{\phi^*}(3) = n$.

If the order of S_2 is $2n$, then $v_{g^*}(1) = v_{g^*}(3) = n$. Hence, $v_{\phi^*}(1) = n-1$ and $v_{\phi^*}(3) = n$ if $g^*(x) = 1$; $v_{\phi^*}(1) = n$ and $v_{\phi^*}(3) = n-1$ if $g^*(x) = 3$.

Suppose the order of S_1 is $2n+1$. Then the order of S_2 is $2n+1$. By conditions 1 and 2, the number of i-edges are n under ϕ for $0 \leq i \leq 3$. In this case $v_{g^*}(\ell) = n+1$ and $v_{g^*}(\ell_1) = n$, where $g^*(x) = \ell$ and $\{\ell, \ell_1\} = \{1,3\}$. Similarly, we have $v_{\phi^*}(2) = v_{\phi^*}(1) = v_{\phi^*}(3) = n$ and $v_{\phi^*}(0) = n+1$. □

Lemma 6. *Let $g : E(T) \to \{1,3\}$ be an edge labeling of tree T of order n, which satisfies the condition (2) of Lemma 5, then $n \not\equiv 2 \pmod 4$.*

Proof. Let E_3 be the set of all 3-edges in T, and let $T_3 = T[E_3]$. Since any 3-vertex must be incident to three edges, all 3-vertices are in T_3. Also, each 3-vertex is of odd degree in T_3. Thus, $v_{g^*}(3)$ is even. Hence, $n = v_{g^*}(3) + v_{g^*}(1) \equiv v_{g^*}(1) \pmod 2$. If $v_{g^*}(1)$ is odd, then $n \not\equiv 2 \pmod 4$. If $v_{g^*}(1)$ is even, then $v_{g^*}(1) = v_{g^*}(3)$. Hence, $n \equiv 0 \pmod 4$. □

By Lemma 5, we have the following corollary.

Corollary 4. *Suppose S_1 and S_2 are the trees such that $V(S_1) \cap V(S_2) = \{x\}$. Let the order of S_1 be $4k+1$ or $4k+2$ and the order of S_2 be $4k+2$, where $k \geq 1$. Let $f : E(S_1) \to \{0,2\}$ and $g : E(S_2) \to \{1,3\}$ be the edge labeling, which satisfy the following conditions:*

(a) $0 \leq e_f(2) - e_f(0) \leq 1$ *and* $0 \leq v_{f^*}(2) - v_{f^*}(0) \leq 1$;

(b) $g^*(x) = \ell$;

(c) $|e_g(1) - e_g(3)| \leq 1$ *and* $v_{g^*}(\ell) - v_{g^*}(\ell_1) = 2$, *where* $\{\ell, \ell_1\} = \{1,3\}$.

Let ϕ be the labeling of $S_1 \cup S_2$ by combining f and g. Then ϕ is an edge 4-product cordial labeling of $S_1 \cup S_2$.

A vertex that satisfies the condition (3) in Lemma 5 or the conditions (b) and (c) in Corollary 4 is called a *major vertex* under g.

Lemma 7. *If $n \not\equiv 2 \pmod 4$ and $n \geq 3$, then there exists a labeling $g : E(P_n) \to \{1,3\}$ that satisfies the condition (2) of Lemma 5. If $n \equiv 2 \pmod 4$ and $n \geq 2$, then there exists a labeling $g : E(P_n) \to \{1,3\}$ that satisfies the condition (c) of Corollary 4.*

Proof. We label the edges of a path P_n by 1,3,3,1 evenly and denote this required labeling by g. □

Lemma 8. *If $m + n \not\equiv 2 \pmod 4$ and $n \geq 3$, $m \geq 1$, then there exists a labeling $g : E(C(n,m)) \to \{1,3\}$ that satisfies the condition (2) of Lemma 5. If $m + n \equiv 2 \pmod 4$ and $n \geq 3$, $m \geq 2$, then there exists a labeling $g : E(C(n,m)) \to \{1,3\}$ that satisfies the condition (c) of Corollary 4. Moreover, u_1 is the major vertex under g.*

Proof. We will define a labeling $g : E(C(,m)) \to \{1,3\}$ by the following approach. We first suitably label the edge $u_n v_i$ for $1 \le i \le m$. And then label the edge of the path $u_n u_{n-1} \cdots u_1$. There are four cases. We put the details in Appendix B. Hence, we have the theorem. □

Now, we consider the comet graph $C(n,m)$, which has $m+1$ leaves. From Theorem 3, we have

$$n \le \begin{cases} 3m+4 & \text{if } m+n \equiv 0 \pmod 4; \\ 3m+5 & \text{if } m+n \equiv 1 \pmod 4; \\ 3m+10 & \text{if } m+n \equiv 2 \pmod 4; \\ 3m+11 & \text{if } m+n \equiv 3 \pmod 4. \end{cases}$$

When $n = 1,2$, we have $C(n,m) \cong K_{1,n+m-1}$, which is a star. It is easy to check that $K_{1,n+m-1}$ is edge 4-product cordial when $n+m \ge 5$.

Theorem 4. *For $n \ge 3$, the comet graph $C(n,m)$ is edge 4-product cordial if and only if*

$$n \le \begin{cases} 3m+4 & \text{if } m+n \equiv 0 \pmod 4; \\ 3m+5 & \text{if } m+n \equiv 1 \pmod 4; \\ 3m+10 & \text{if } m+n \equiv 2 \pmod 4; \\ 3m+11 & \text{if } m+n \equiv 3 \pmod 4. \end{cases}$$

Proof. The necessary part is shown in the discussion above. Now we have to show the sufficient part. Let $N = \lfloor \frac{m+n}{4} \rfloor$. We can check that $N \le m+1$ when $m+n \equiv 0,1 \pmod 4$; and $N \le m+2$ when $n+m \equiv 2,3 \pmod 4$.

We split the graph $C(n,m)$ into two subgraphs, S_1 and S_2, with a common vertex x. Then, we define the labelings f and g for S_1 and S_2, respectively, such that f and g satisfy all the conditions of Lemma 5 or Corollary 4. The details are referred to Appendix C.

Hence, we have the theorem. □

Example 4. *Edge 4-product labelings for $C(9,2)$, $C(8,3)$, and $C(5,6)$ are shown in Figure 4.*

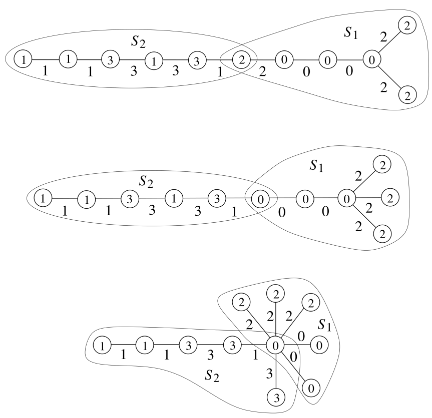

Figure 4. Edge 4-product labelings for $C(9,2)$, $C(8,3)$, and $C(5,6)$.

Example 5. *Edge 4-product labelings for $C(10,2)$, and $C(5,7)$ are provided in Figure 5.*

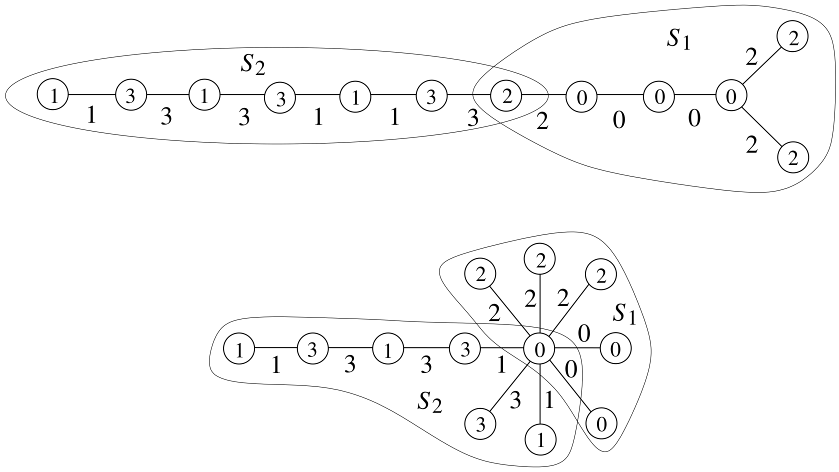

Figure 5. Edge 4-product labelings for $C(10, 2)$ and $C(5, 7)$.

Example 6. *Edge* 4*-product labelings for* $C(16, 2)$ *and* $C(17, 2)$ *are shown in Figure 6.*

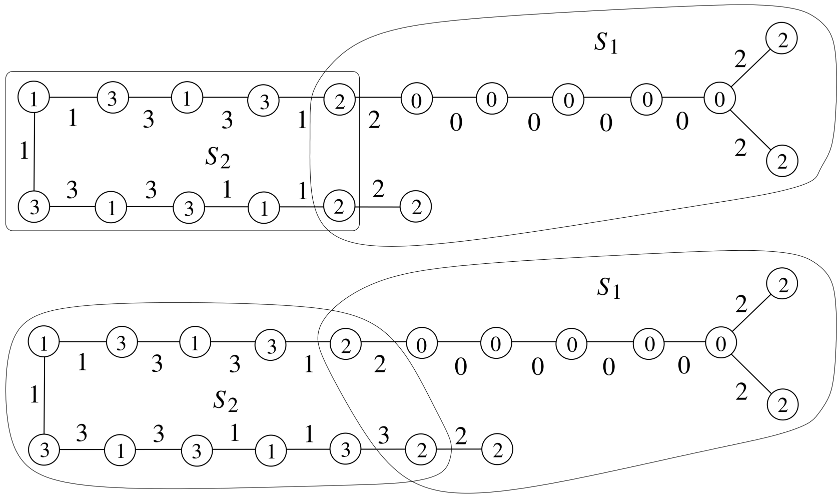

Figure 6. Edge 4-product labelings for $C(16, 2)$ and $C(17, 2)$.

Remark 3. *Suppose T is a tree of order* $4N + k$*, which admits an edge* 4*-product cordial labeling* f*, where* $0 \leq k \leq 3$*. Let* E_i *be the set of edges labeled by* i*, where* $i = 0, 2$ *and* $H = T[E_2 \cup E_0]$*. By Theorem 3 we obtain the following results.*

(A) *If* $k = 0, 1$*, then* $q = N$*. Thus, H is a tree that has at least N leaves.*

(B) *If* $k = 2, 3$*, then* $q \geq N - 1$*. Clearly,* $q \leq N + 1$*. Also, we have* $e_f(0) = N$ *and* $v_{f^*}(0) = N + 1$*.*

- *B1.* *Suppose* $q = N + 1$*. Clearly,* $e_f(2) = N + 1$ *and* $v_{f^*}(2) = N + 1$*. Then H is a tree of order* $2N + 2$ *that has at least* $N + 1$ *leaves.*
- *B2.* *Suppose* $q = N$*. Recall that* $q = 2e_f(2) - v_{f^*}(2)$*. Since* $v_{f^*}(2) \leq N + 1$*, we have* $e_f(2) \leq N + \frac{1}{2}$*. Thus,* $e_f(2) = N$ *and* $v_{f^*}(2) = N$*. Then H is a tree of order* $2N + 1$ *that has at least N leaves.*
- *B3.* *Suppose* $q = N - 1$*. Since* $v_{f^*}(2) \leq N + 1$*, we have* $e_f(2) = N$ *and* $v_{f^*}(2) = N + 1$*. Then H is a disjoint union of a tree* T' *of order* $2N$ *with* P_2*. Moreover,* T' *has at least* $N - 1$ *leaves.*

Theorem 5. *Suppose* $n \geq 2$*,* $M \geq m \geq 2$ *and* $M + m + n = 4N + k$*, where* $0 \leq k \leq 3$*.*

A. *When* $k = 0, 1$*. The graph* $DC(n, M, m)$ *is an edge* 4*-product cordial if and only if* $M \geq N - 1$*.*

B. *When $k = 2, 3$. The graph $DC(n, M, m)$ is an edge 4-product cordial if and only if $M \geq N - 2$.*

Proof. Suppose there is an edge 4-product cordial labeling f for $T = DC(n, M, m)$. Let H be the edge-induced subgraph defined in Remark 3.

(A) Suppose $k = 0, 1$. By Remark 3, H is a tree of order $2N + k$ that has at least N leaves. Suppose $m \leq M \leq N - 2$. Then $N = \frac{M+m+n-k}{4} \leq \frac{2M+n-k}{4} \leq \frac{2N-4+n-k}{4}$. Thus, $n \geq 2N + k + 4 \geq 2N + 4$. Since H has at least N leaves, we have $H \cong DC(n, M_1, m_1)$, where $M_1 + m_1 \geq N$. But the order of H is at least $3N + 4$, which is a contradiction. Thus, if $M \leq N - 2$, then $DC(n, M, m)$ is not an edge 4-product cordial graph.

(B) Suppose $k = 2, 3$. Then $e_f(0) = N$ and $v_{f^*}(0) = N + 1$.

1. Suppose $q = N + 1$. Then H is a subtree of $DC(n, M, m)$ of order $2N + 2$ that has $N + 1$ leaves. Suppose $m \leq M \leq N - 1$. Similarly to Case A, we obtain a contradiction. Thus, if $M \leq N - 1$, then $DC(n, M, m)$ is not an edge 4-product cordial graph.
2. Suppose $q = N$. Then H is a subtree of $DC(n, M, m)$ of order $2N + 1$ that has N leaves. Suppose $m \leq M \leq N - 2$. Similarly to Case A, we obtain a contradiction. Thus, if $M \leq N - 2$, then $DC(n, M, m)$ is not an edge 4-product cordial graph.
3. Suppose $q = N - 1$. Then H is a disjoint union of a tree T' of order $2N$ with P_2. Moreover, T' has at least $N - 1$ leaves. Thus, T' must be a comet $C(n_1, m_1)$ such that $n_1 + m_1 = 2N$ and $m_1 \geq N - 2$. Thus, if $M \leq N - 3$, then $DC(n, M, m)$ is not an edge 4-product cordial.

Consequently, if $DC(n, M, m)$ is an edge 4-product cordial, then $M \geq N - 1$ when $n + M + m \equiv 0, 1 \pmod 4$; and $M \geq N - 2$ when $n + M + m \equiv 2, 3 \pmod 4$.

For the sufficient part, we split the graph $DC(n, M, m)$ into two subgraphs S_1 and S_2 with one or two common vertices. We label S_1 by 0 and 2, and S_2 by 1 and 3, respectively, such that these labelings induce an edge 4-product cordial labeling for $DC(n, M, m)$.

Since $M, n \geq 2$, we have $M + n \geq \frac{M+m+n}{2} + \frac{n}{2} \geq \frac{4N+k}{2} + 1 = 2N + 1 + \frac{k}{2}$. This guarantees that the comets S_1 and S_2 defined below are well-defined. The details are referred to Appendix D.

This completes the proof. □

Example 7. *Edge 4-product labelings for $DC(14, 3, 2)$ and $DC(15, 2, 2)$ are shown in Figure 7.*

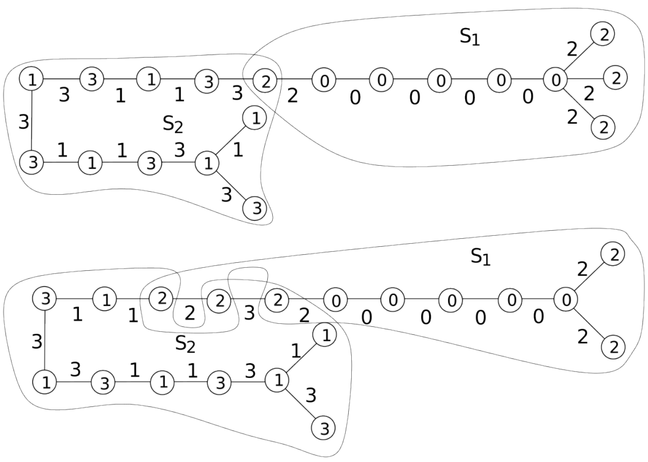

Figure 7. Edge 4-product labelings for $DC(14, 3, 2)$ and $DC(15, 2, 2)$.

5. Edge 5-Product Cordial Trees

In order to prove the main theorems, first we prove the following lemma. Note that, by Lemma 2, the path P_5 is not an edge 5-product cordial.

Lemma 9. *The path graph P_n is an edge 5-product cordial for $n \geq 3$ and $n \neq 5$.*

Proof. We define an edge labeling f_n for P_n recurrently, and this labeling is represented by $(a_1, \ldots, a_{n-1})$ if $f_n(u_i u_{i+1}) = a_i$, $1 \leq i \leq n-1$.

We present the edge 5-product cordial labeling for P_n, $3 \leq n \leq 12$, except $n = 5$ in Table 3.

Table 3. Edge 5-product cordial labeling for P_n, $3 \leq n \leq 12$, except $n = 5$.

n	f_n	$v_{f_n^*}(0)$	$v_{f_n^*}(1)$	$v_{f_n^*}(2)$	$v_{f_n^*}(3)$	$v_{f_n^*}(4)$
3	(3,2)	0	1	1	1	0
4	(1,2,4)	0	1	1	1	1
6	(3,2,1,4,0)	2	1	1	1	1
7	(1,3,2,1,4,0)	2	2	1	1	1
8	(1,1,3,3,4,2,0)	2	2	1	2	1
9	(1,2,4,1,4,3,2,0)	2	2	2	1	2
10	(1,3,3,4,4,2,2,1,0)	2	2	2	2	2
11	(1,3,3,4,4,2,2,1,0,0)	3	2	2	2	2
12	(1,3,3,4,4,2,2,1,1,0,0)	3	3	2	2	2

For $n \geq 13$, we define f_n recurrently as $f_n = f_{m-i+10} = (1,3,3,4,4,2,2,1,a_1,\ldots,a_{m-i-1},0,0)$, where $0 \leq i \leq 9$ and $m \geq 12$. When $m - i \geq 7$, we have $a_1 = 1$ and $a_{m-i-1} = 0$. Thus, $e_{f_{m-i+10}}(j) = e_{f_{m-i}}(j) + 2$, $v_{f^*_{m-i+10}}(j) = v_{f^*_{m-i}}(j) + 2$, $0 \leq j \leq 4$.

For $m - i = 3$, $v_{f_{13}^*}(0) = 3$, $v_{f_{13}^*}(1) = 3$, $v_{f_{13}^*}(2) = 2$, $v_{f_{13}^*}(3) = 3$, $v_{f_{13}^*}(4) = 2$.
For $m - i = 4$, $v_{f_{14}^*}(0) = 3$, $v_{f_{14}^*}(1) = 3$, $v_{f_{14}^*}(2) = 3$, $v_{f_{14}^*}(3) = 3$, $v_{f_{14}^*}(4) = 2$.
For $m - i = 5$, $v_{f_{15}^*}(j) = 3$ for all $0 \leq j \leq 4$.
For $m - i = 6$, $v_{f_{16}^*}(0) = 4$ and $v_{f_{16}^*}(j) = 3$ for all $1 \leq j \leq 4$.

Thus, f_n is an edge 5-product cordial labeling of P_n for $n \geq 6$. Moreover, f_3 and f_4 are the required labelings for P_3 and P_4, respectively.

This completes the proof. □

Theorem 6. *The comet graph $C(n,m)$ is an edge 5-product cordial for $n \geq 3$ and $m \geq 2$, except $(n,m) = (3,2)$.*

Proof. Let $m = 5k + r$, where $0 \leq r \leq 4$. In order to obtain an edge 5-product cordial labeling of $C(n,m)$ for $n \geq 3$ and $m \geq 2$, except for $(n,m) = (3,2)$, we split $C(n,m)$ into two subgraphs, $K_{1,5k}$ and $C(n,r)$, with a common vertex u_n. Note that when $k = 0$, $K_{1,5k}$ does not appear; when $r = 0$, $C(n,r) \cong P_n$. For the last case, it has been proved in Lemma 9.

First, we label P_n by using f_n, which is defined below. For $1 \leq r \leq 4$, we label $K_{1,r}$ to balance the number of i-edges and i-vertices, as shown in Table 4.

For $n \geq 13$, we label $f_n = f_{m+10} = (1,3,3,4,4,2,2,1,f_m,0,0)$, where $m \geq 3$. Then the difference between the number of i-edges and j-vertices does not change for all $0 \leq i,j \leq 4$. Hence, according to the table above, we have an edge 5-product cordial labeling for $C(n,r)$, where $n \geq 3$ and $1 \leq r \leq 4$.

For $k \geq 1$, we label the edges of $K_{1,5k}$ by 0, 1, 2, 3, and 4 evenly and denote this labeling by ϕ. Thus, ϕ is an edge 5-product labeling for $C(n, 5k+r)$ for $n \geq 3$ and $5k + r \geq 2$, except for $(n,m) = (3,2)$. By Corollary 2, $C(3,2)$ is not an edge 5-product cordial graph. This completes the proof. □

Table 4. Edge labeling for P_n, $3 \leq n \leq 12$ and $K_{1,r}$, $1 \leq r \leq 4$.

n	f_n	Priority for Numbers Added to $K_{1,r}$
3	(1,3)	4,2,0,1
4	(3,2,1)	4,0,1,2
5	(3,2,1,4)	0,1,2,3
6	(3,2,1,4,0)	1,2,3,4
7	(1,3,2,1,4,0)	2,3,4,0
8	(1,1,3,3,4,2,0)	2,4,0,3
9	(1,2,4,1,4,3,2,0)	3,0,1,2
10	(1,3,3,4,4,2,2,1,0)	0,1,2,3
11	(1,3,3,4,4,2,2,1,0,0)	1,2,3,4
12	(1,3,3,4,4,2,2,1,1,0,0)	2,3,4,0

Example 8. *For the comet $C(10,13)$, we separate it into two edge-disjoint graphs, $C(10,3)$ and $K_{1,10}$. From the table above, we label P_{10} as 1, 3, 3, 4, 4, 2, 2, 1, 0, and label the edges of $K_{1,3}$ by 0, 1, and 2. The resulting labeling is an edge labeling for $C(10,3)$. The numbers of 1-and 2-edges are 3, and the numbers of 0-, 3- and 4-edges are 2. The numbers of 0-, 1-, and 2- vertices are 3, and the numbers of 3- and 4- vertices are 2.*

Finally, we label the edges of $K_{1,10}$ evenly by $0,1,2,3,4$. We obtain three 0-vertices and two i-vertices, where $1 \leq i \leq 4$. The centers of $K_{1,10}$ and $K_{1,3}$ will be merged; thus, we obtain an edge 5-product cordial labeling ϕ for $C(10,13)$. We can check that $e_\phi(0) = 4$, $e_\phi(1) = 5$, $e_\phi(2) = 5$, $e_\phi(3) = 4$, $e_\phi(4) = 4$; $v_{\phi^*}(0) = 5$, $v_{\phi^*}(1) = 5$, $v_{\phi^*}(2) = 5$, $v_{\phi^*}(3) = 4$, $v_{\phi^*}(4) = 4$.

Theorem 7. *The double comet graph $DC(n,M,m)$ is an edge 5-product cordial for $n \geq 2$ and $M \geq m \geq 2$.*

Proof. Let $S_1 \cong K_{1,M}$ with center u_n and $S_2 \cong K_{1,m}$ with center u_1. Then $DC(n,M,m) = P_n \cup S_1 \cup S_2$.

We define an edge labeling α for S_2 and the selected edges of S_1 by 0,1,2,3,4 evenly. Consider $m = 3p + q$, where $0 \leq q \leq 2$. First, we assume $p \geq 1$.

1. For $q = 0$, we label the edges of S_2 by 1, 2, 3 evenly and p, p edges of S_1 by $0, 4$, respectively.
2. For $q = 1$, we label $p-1, p, p, 2$ edges of S_2 by $1,2,3,4$, respectively, and $p, p-2, 1$ edges of S_1 by $0,4,1$, respectively.
3. For $q = 2$, we label $p, p, p, 2$ edges of S_2 by $1,2,3,4$, respectively, and $p, p-2$ edges of S_1 by $0,4$, respectively.

Note that $\alpha^*(u_1) = 1$. Also $e_\alpha(i) = p$ for all i and $v_{\alpha^*}(0) - 1 = v_{\alpha^*}(1) - 1 = v_{\alpha^*}(j)$ for $2 \leq j \leq 4$.

Now the unlabeled edges form $C(n, M - 2p + \epsilon)$, where

$$\epsilon = \begin{cases} 0 & \text{if } q = 0, \\ 1 & \text{if } q = 1, \\ 2 & \text{if } q = 2. \end{cases}$$

Hence, $M - 2p + \epsilon \geq p + q + \epsilon \geq 2$ except for $M = m = 3$. By Theorem 6, there is an edge 5-product cordial labeling for $C(n, M - 2p + \epsilon)$ for $M - 2p + \epsilon \geq 2$.

For $M = m = 3$, $C(n,1) \cong P_{n+1} = u_1 \cdots u_n v_3$. We label this P_{n+1} by the labeling f_{n+1} defined in Lemma 9. Now we check the number of i-vertices.

Before labeling P_{n+1}, we have $\alpha^*(v_1) = 0 = \alpha^*(u_n)$, $\alpha^*(v_2) = 4$, $\alpha^*(w_1) = 1$, $\alpha^*(w_2) = 2$, $\alpha^*(w_3) = 3$ and $\alpha^*(u_1) = 1$. Suppose $n+1 \geq 7$. After labeling P_{n+1}, the vertex u_n is still 0-vertex and the vertex u_1 changes from 1-vertex to $f^*_{n+1}(u_1)$. Thus $\alpha^*(u_n) = 0$

and $\alpha^*(u_1) = 1$ do not count towards the number of 0-vertices and 1-vertices. Thus, the combined labeling is an edge 5-product cordial labeling for $DC(n,3,3)$.

For $n+1 = 3,4,5$, the required labeling is shown in the following example.

Suppose $m = 2$. If $M \geq 3$, then $\alpha(u_1w_1) = 2$, $\alpha(u_1w_2) = 3$, $\alpha(u_nv_1) = 0$, $\alpha(u_nv_2) = 1$ and $\alpha(u_1v_3) = 4$. The unlabeled edges form $C(n, M-3)$. The argument is similar to the cases above. If $M = 2$, let $\alpha(u_1w_1) = 2$, $\alpha(u_1w_2) = 3$, $\alpha(u_nv_1) = 1$, $\alpha(u_nv_2) = 4$ and $\alpha(u_1u_3) = 0$. The unlabeled edges form P_{n-1}. If $n = 2$, then we have an edge 5-product cordial labeling for $DC(2,2,2)$. If $n = 3$, then label u_1u_2 by 1. We have an edge 5-product cordial labeling for $DC(3,2,2)$. If $n \geq 4$, then the labeling is the same as $M = m = 3$. □

Example 9. *An edge* 5*-product cordial labelings for* $DC(n,3,3)$*, where* $n = 2,3,4,5$ *are shown in Figure 8.*

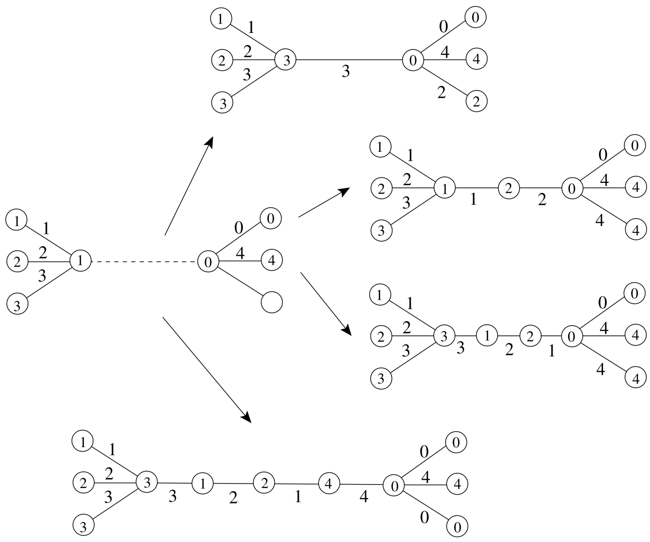

Figure 8. Edge 5-product labelings for $DC(2,3,3)$, $DC(3,3,3)$, $DC(4,3,3)$, and $DC(5,3,3)$.

6. Conclusions

The notion of edge k-product cordial labeling was introduced only in the year 2025, and the authors showed that star, bistar, and path unions of star graphs admit edge k-product cordial labeling. They also investigated the edge k-product cordial behavior of the shadow and the splitting graph of a star. In this work, we further explore the relationship between the number of edges and vertices labeled with 0 in edge k-product cordial trees and investigate the edge k- product cordiality of trees of order k. Also, we establish the edge k-product cordial properties of comet and double comet trees for $k = 3,4$, and 5. It is noted that edge k-product cordial labeling is a recent concept, and only a limited study has been carried out. Future researchers have ample scope to identify the families of graphs that admit or do not admit edge k-product cordial labeling, and also to investigate the edge k-product cordial behavior of a larger number of standard graphs.

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Appendix A. The Details About the Labelings for the Proof of Theorem 2

1. When $q = 2$, $1 \geq M - 2p - 4 \geq (4p + 2) - 2p - 4 = 2p - 2$, which implies $p = 0, 1$. That is, $m = 2, 6$.
   If $m = 2$ and $M - 4 \leq 1$, then $M = 2, 3, 4, 5$.
   (i) Suppose $M = 2$ and $n \geq 6$. Let $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_nv_1) = \alpha(u_nv_2) = 2$ and $\alpha(u_nu_{n-1}) = \alpha(u_{n-2}u_{n-1}) = 0$. Then $\alpha^*(u_{n-2}) = 0$. Also, $v_{\alpha^*}(0) - 1 = v_{\alpha^*}(1) = v_{\alpha^*}(2) = 2$. We form a path P_{n-2} from the unlabeled edges. We combine α and f_{n-2} to get an edge 3-product cordial labeling for $DC(n, 2, 2)$, where f_{n-2} is defined in the proof of Lemma 3. For $2 \leq n \leq 5$, the required labeling is shown in Figure A1.

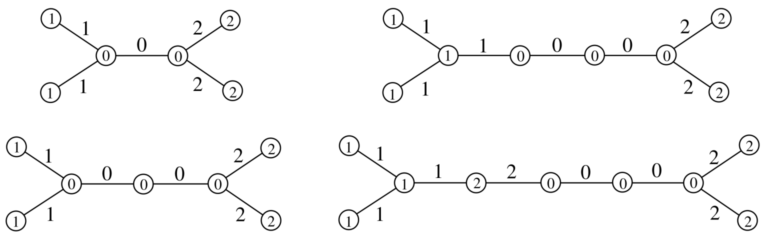

Figure A1. Edge 3-product cordial labelings of $DC(2, 2, 2)$, $DC(3, 2, 2)$, $DC(4, 2, 2)$, and $DC(5, 2, 2)$.

   (ii) Suppose $M = 3$ and $n \geq 5$. Let $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_nv_1) = \alpha(u_nv_2) = 2$, $\alpha(u_nv_3) = 0$ and $\alpha(u_nu_{n-1}) = 0$. Then $\alpha^*(u_{n-1}) = 0$. Also, $v_{\alpha^*}(0) - 1 = v_{\alpha^*}(1) = v_{\alpha^*}(2) = 2$. We form a path P_{n-1} from the unlabeled edges. Combine α and f_{n-1} to get an edge 3-product cordial labeling for $DC(n, 3, 2)$, where f_{n-1} is defined in the proof of Lemma 3. For $2 \leq n \leq 4$, the required labeling is shown in Figure A2.

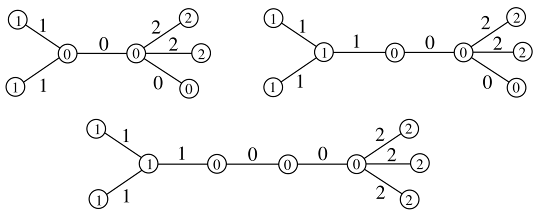

Figure A2. Edge 3-product cordial labeling of $DC(2, 3, 2)$, $DC(3, 3, 2)$, and $DC(4, 3, 2)$.

   (iii) Suppose $M = 4$. If $n \geq 4$, then by Remark 1, $H \cong C(n, M - 4)$ admits an edge 3-product cordial labeling. So we have to consider only $n = 2, 3$. Let $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_nv_1) = \alpha(u_nv_2) = 2$, $\alpha(u_nv_3) = \alpha(u_nv_4) = 0$. Also, if $n = 2$, then $\alpha(u_1u_2) = 1$ and if $n = 3$, then $\alpha(u_1u_2) = 1$, $\alpha(u_2u_3) = 2$. Hence, α is an edge 3-product labeling for $DC(n, 4, 2)$.
   (iv) Suppose $M = 5$. If $n \geq 3$, then by Remark 1, $H \cong C(n, M - 4)$ admits an edge 3-product cordial labeling. So we have to consider only $n = 2$. Let $\alpha(u_1w_1) = \alpha(u_1w_2) = 1$, $\alpha(u_nv_1) = \alpha(u_nv_2) = \alpha(u_nv_3) = 2$, $\alpha(u_nv_4) = \alpha(u_nv_5) = 0$, $\alpha(u_1u_2) = 1$. Hence, α is an edge 3-product labeling for $DC(2, 5, 2)$.

   When $m = 6$, $H \cong C(n, M - 6)$. If $M \geq 8$, then similar to the above case, $DC(n, M, 6)$ is an edge 3-product cordial. Therefore, we must consider only $M = 6, 7$.
   (i) Suppose $M = 6$. We have to consider only $n = 2, 3$. By a similar method for labeling $DC(n, 4, 2)$, we obtain an edge 3-product labeling for $DC(n, 6, 6)$.

(ii) Suppose $M = 7$. We have to consider only $n = 2$. By a similar method for labeling $DC(2,5,2)$, we obtain an edge 3-product labeling for $DC(2,7,6)$.

2. When $q = 3$, we have $1 \geq M - 2p - 3 \geq (4p+3) - 2p - 3 = 2p$. This implies $p = 0$ that is, $m = 3$. Now, $H \cong C(n, M-3)$. We have to consider only $M = 3, 4$.

   (i) Suppose $M = 3$. We have to consider only $n = 2, 3$. By a similar method for labeling $DC(n,4,2)$, we obtain an edge 3-product labeling for $DC(n,3,3)$.

   (ii) Suppose $M = 4$. We have to consider only $n = 2$. By a similar method for labeling $DC(2,7,6)$, we obtain an edge 3-product labeling for $DC(2,4,3)$.

3. When $q = 4$, we have $1 \geq M - 2p - 2 \geq (4p+4) - 2p - 2 = 2p + 2$, which is not possible.
4. When $q = 5$, we have $1 \geq M - 2p - 4 \geq (4p+5) - 2p - 4 = 2p + 1$. This implies $p = 0$ that is, $m = 5$. Now $H \cong C(n, M-4)$. We need to consider only $M = 5$ and $n = 2$. By a similar method for labeling $DC(2,5,2)$, we obtain an edge 3-product labeling for $DC(2,5,5)$.

Appendix B. The Details About the Labelings for the Proof of Lemma 8

We define the labeling $g : E(C(n,m)) \to \{1,3\}$ as follows:

1. Suppose $m = 4k \geq 4$. We label $u_n v_i$ by 1 for $1 \leq i \leq 2k$ and $u_n v_i$ by 3 for $2k+1 \leq i \leq 4k$. Here, the induced vertex label for u_n is 1. Also, we label $u_n \cdots u_1$ by 1, 3, 3, 1 evenly.
   If $n \equiv 0 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3)$.
   If $n \equiv 1 \pmod 4$, then $g^*(u_1) = 1$ and $v_{g^*}(1) - 1 = v_{g^*}(3)$.
   If $n \equiv 3 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3) - 1$.
   If $n \equiv 2 \pmod 4$, then $g^*(u_n) = 1 = g^*(u_1)$ and $v_{g^*}(1) - 2 = v_{g^*}(3)$.
2. Suppose $m = 4k+1 \geq 1$. If $k \geq 1$, we label $u_n v_i$ by 1 for $1 \leq i \leq 2k$ and $u_n v_i$ by 3 for $2k+1 \leq i \leq 4k$. Here, the induced vertex label for u_n is 1 (if $k \geq 1$) and an edge $u_n v_m$ is not labeled. Label the path $v_m u_n \cdots u_1$ by 1, 3, 3, 1 evenly. Here, $g^*(u_n) = 3$.
   If $n \equiv 3 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3)$.
   If $n \equiv 2 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3) - 1$.
   If $n \equiv 0 \pmod 4$, then $g^*(u_1) = 1$ and $v_{g^*}(1) - 1 = v_{g^*}(3)$.
   If $n \equiv 1 \pmod 4$, then $g^*(u_1) = 1$ and $v_{g^*}(1) - 2 = v_{g^*}(3)$.
3. Suppose $m = 4k+2 \geq 2$. We label $u_n v_i$ by 1 for $1 \leq i \leq 2k+1$ and $u_n v_i$ by 3 for $2k+2 \leq i \leq 4k+2$; and label $u_n u_{n-1}$ and $u_{n-1} u_{n-2}$ by 3 and 1, respectively. Also, we label the path $u_{n-2} \cdots u_1$ by 1, 3, 3, 1 evenly. Here, the induced vertex label for u_n and u_{n-2} are 1.
   If $n \equiv 3 \pmod 4$, then $g^*(u_1) = 1$ and $v_{g^*}(1) - 1 = v_{g^*}(3)$.
   If $n \equiv 1 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3) - 1$.
   If $n \equiv 2 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3)$.
   If $n \equiv 0 \pmod 4$, then $g^*(u_n) = 1 = g^*(u_1)$ and $v_{g^*}(1) - 2 = v_{g^*}(3)$.
4. Suppose $m = 4k+3 \geq 3$. We label $u_n v_i$ by 1 for $1 \leq i \leq 2k+2$, $u_n v_i$ by 3 for $2k+3 \leq i \leq 4k+3$ and $u_n u_{n-1}$ by 3. Then $g^*(u_n) = 1$ and $g^*(u_{n-1}) = 3$. We label the path $u_{n-1} \cdots u_1$ by 1, 3, 3, 1 evenly.
   If $n \equiv 0 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3) - 1$.
   If $n \equiv 1 \pmod 4$, then $g^*(u_1) = 3$ and $v_{g^*}(1) = v_{g^*}(3)$.
   If $n \equiv 2 \pmod 4$, then $g^*(u_1) = 1$ and $v_{g^*}(1) - 1 = v_{g^*}(3)$.
   If $n \equiv 3 \pmod 4$, then $g^*(u_n) = 1 = g^*(u_1)$ and $v_{g^*}(1) - 2 = v_{g^*}(3)$.

Appendix C. The Details About the Labelings for the Proof of Theorem 4

1. Suppose $m + n = 4N$. When $m < 2N - 1$, we split $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N, m)$ and $S_2 = P_{2N+1}$ with a common vertex u_{2N+1}. Note that, since $N \leq m + 1$, S_1 has at least N leaves. When $m \geq 2N - 1$, we split $C(n,m)$ into two subgraphs, namely $S_2 = C(n, m - 2N + 1)$ and $S_1 = K_{1,2N-1}$ with a common vertex u_n. Recall that $C(n,0) \cong P_n$ and $C(n,1) \cong P_{n+1}$.
   We label N pendant edges of S_1 by 2 and the remaining $N - 1$ edges by 0 and denote this labeling by f. Then $e_f(2) = e_f(0) + 1 = N$, $v_{f^*}(2) = v_{f^*}(0) = N$ and $f^*(u_{2N+1}) = 0$.
   By Lemmas 7 and 8, we have a labeling g of S_2, which satisfies the conditions (2) and (3) of Lemma 5.
2. Suppose $m + n = 4N + 1$. When $m < 2N$, we split $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N, m)$ and $S_2 = P_{2N+1}$ with a common vertex u_{2N+1}. Note that, since $N \leq m + 1$, S_1 has at least N leaves. When $m \geq 2N$, we split $C(n,m)$ into two subgraphs, namely $S_2 = C(n, m - 2N)$ and $S_1 = K_{1,2N}$ with a common vertex u_n. Similar to Case 1, we have the labelings f and g of S_1 and S_2, respectively, which satisfies all the conditions of Lemma 5.
3. Suppose $m + n = 4N + 2$. Then, $N \leq m + 2$. First, we assume $N \leq m + 1$. When $m < 2N$, we split $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N - 1, m)$ and $S_2 = P_{2N+2}$ with a common vertex u_{2N+2}. Note that, since $N \leq m + 1$, S_1 has at least N leaves. When $m \geq 2N$, we split the graph $C(n,m)$ into two subgraphs, namely $S_2 = C(n, m - 2N)$ and $S_1 = K_{1,2N}$ with a common vertex u_n.
   We label N pendant edges of S_1 by 2 and the remaining N edges by 0. By Lemmas 7 and 8, we have a labeling g of S_2 satisfying the condition s (2) and (3) of Lemma 5 or the conditions (b) and (c) of Corollary 4.
   Suppose $N = m + 2$. we split the graph $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N - 2, m) \cup P_2$ and $S_2 = u_2u_3 \cdots u_{2N+3} \cong P_{2N+2}$ such that $V(S_1) \cap V(S_2) = \{u_{2N+3}, u_2\}$, where $P_2 = u_1u_2$. Now label N pendant edges of S_1 by 2 and the remaining N edges by 0. Also, label the edges of S_2 by 1,3,3,1 evenly, denoted by g. Finally, we have u_2 and u_{2N+3} are 2-vertices. Consequently, the number of 0-vertices and 2-vertices are $N + 1$ and those of 1-vertices and 3-vertices are N.
4. Suppose $m + n = 4N + 3$. Then, $N \leq m + 2$. First, we assume $N \leq m + 1$. When $m < 2N$, we split $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N - 2, m)$ and $S_2 = P_{2N+3}$ with a common vertex u_{2N+3}. When $m \geq 2N$, we split $C(n,m)$ into two subgraphs, namely $S_2 = C(n, m - 2N)$ and $S_1 = K_{1,2N}$ with a common vertex u_n. Note that, the order of S_1 is $2N + 1$. We label N pendant edges of S_1 by 2 and the remaining N edges by 0. By Lemmas 7 and 8, we have a labeling g of S_2 satisfying the conditions (2) and (3) of Lemma 5 or the conditions (b) and (c) of Corollary 4. We can check that the number of 0-vertices is $N + 1$ and the number of 2-vertices is N.
   Suppose $N = m + 2$. We split $C(n,m)$ into two subgraphs, namely $S_1 = C(n - 2N - 3, m) \cup P_2$ and $S_2 = u_2u_3 \cdots u_{2N+4} \cong P_{2N+3}$ such that $V(S_1) \cap V(S_2) = \{u_{2N+4}, u_2\}$, where $P_2 = u_1u_2$. Now label N pendant edges of S_1 by 2 and the remaining N edges by 0. Also, label the edges of S_2 by 1,3,3,1 evenly, denoted by g. Finally, we have u_2 and u_{2N+4} are 2-vertices. Consequently, the number of 0-vertices and 2-vertices are $N + 1$; and the number of 1-vertices and 3-vertices are either N or $N + 1$ and not both.

Appendix D. The Details About the Labelings for the Proof of the Sufficient Part of Theorem 5

1. Suppose $k = 0, 1$. We assume that $M \geq N - 1$.

1a. Suppose $M + m + n = 4N$. We split the graph $DC(n, M, m)$ into two subgraphs, namely $S_1 = C(2N - M, M)$ and $S_2 = C(2N - m + 1, m)$ with a common vertex u_{2N-m+1}. Note that the order of S_1 and S_2 are $2N$ and $2N + 1$, respectively.
We label N pendant edges of S_1 by 2 and the other edges of S_1 by 0. Consequently, we obtain N 2-vertices and N 0-vertices. By Lemma 8, we have a labeling g for S_2 such that u_{2N-m+1} is the major vertex. Combine these two labelings to get the required labeling for $DC(n, M, m)$.

1b. Suppose $M + m + n = 4N + 1$. We split $DC(n, M, m)$ into two subgraphs, namely $S_1 = C(2N - M + 1, M)$ and $S_2 = C(2N - m + 1, m)$ with a common vertex u_{2N-m+1}. Note that the order of S_1 and S_2 are $2N + 1$. Similarly to the Case 1a, we have the required labeling for $DC(n, M, m)$.

2. Suppose $k = 2, 3$. Now, we assume that $M \geq N - 2$.

2a. Suppose $M + m + n = 4N + 2$.
If $M \geq N - 1$, then we split $DC(n, M, m)$ into two subgraphs, namely $S_1 = C(2N - M + 1, M)$ and $S_2 = C(2N - m + 2, m)$ with a common vertex u_{2N-m+2}. Note that the orders of S_1 and S_2 are $2N + 1$ and $2N + 2$. Similarly to Case 1a, we get the required labeling for $DC(n, M, m)$.
If $M = N - 2$, let S_1 be the disjoint union of $C(2N - M, M)$ with $P_2 = u_{2N-m}u_{2N-m+1}$ and S_2 is the disjoint union of $C(2N - m, m)$ with $P_2 = u_{2N-m+1}u_{2N-m+2}$.
We label all the N pendant edges in S_1 by 2 and the other N edges by 0. Here, the induced labels of u_{2N-m+2}, u_{2N-m+1} and u_{2N-m} are 2. By Lemma 8, we have a labeling for $C(2N - m, m)$ such that u_{2N-m} is the major vertex. Now we label the edge $u_{2N-m+1}u_{2N-m+2}$ by 1 or 3 to have the edge labels evenly. Hence, we obtain the required labeling.

2b. Suppose $M + m + n = 4N + 3$.
If $M \geq N - 1$, then we split $DC(n, M, m)$ into two subgraphs, namely $S_1 = C(2N - M + 1, M)$ and $S_2 = C(2N - m + 3, m)$ with a common vertex u_{2N-m+3}. Note that the orders of S_1 and S_2 are $2N + 1$ and $2N + 3$, respectively. We label N pendant edges of S_1 by 2 and the other edges by 0. By Lemma 8, we have a labeling such that u_{2N-m+3} is the major vertex. We combine these two labelings to obtain the required labeling for $DC(n, M, m)$.
If $M = N - 2$, let S_1 be the disjoint union of $C(2N - M, M)$ with $P_2 = u_{2N-m+2}u_{2N-m+3}$ and S_2 is the disjoint union of $C(2N - m + 2, m)$ with $P_2 = u_{2N-m+3}u_{2N-m+4}$. Similarly to Case 2a, we obtain the required labeling for $DC(n, M, m)$.

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